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A Two Sector Model with Several Externalities and Their Effects on an Urban Setting

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In the first part of this paper, we construct a model of a city with two sectors, residential and production, located respectively in the residential ring and central business district (CBD). We then determine the general equilibrium and the Pareto optimum solution of resource allocation in the city.

Many urban economists from Mills in 1967 to Stull and Henderson in 1974 have designed two sector models. These models have used some simplifying assumption to compensate for their broad scope. For example, Mills' model does not include congestion. Henderson collapses the entire spatial structure into lots and Stull's model has an inelastic demand for housing, and lacking roads, cannot deal with the problem of congestion. In this model (in order to concentrate on the residential sector), we use an aggregate production function for all industry located in the CBD. However, this aggregate production function can be easily broken down into the individual local production processes which generate it.

We include most of the important components of the inner structure of the residential ring, for example, a Muth-type utility function and roads subject to congestion, so that we can investigate in detail the land use pattern and the rent function within the residential ring.

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We also investigate the relations between the production and residential sectors.

The main result of this part of the paper is identifying the externality effect of CBD size. This externality occurs because competition passes the burden of commuting costs inside the CBD from the laborer to his employer. Accordingly, an expansion of the CBD makes it more accessible to the residential ring. Since there is no competitive market to internalize this effect it becomes an externality, a rare case of a positive one with the nature of a public good.

This model, being one of the most general of those which deal with congestion, enables us to update and integrate the recent findings on this subject.

Mills and de Feranti [10] and Solow and Vickrey [13] first introduced congestion into the urban-land-use literature in optimization models, thereby neglecting the externality effect of resource allocation distortion. At the same time, Hochman and Pines [6] developed an equilibrium model with congestion. However, in their paper congestion appeared in an internalized form and thus was not an externality. Oron, Pines, and Sheshinski [12] extended the Hochman-Pines congestion concept to identify the distortion caused by its externality effect and to determine the optimum congestion tolls. They were, however, able to determine only partially the distortion from congestion. Hochman [5] solved a specific model of Mills and de Feranti to show the full distortion pattern of the congestion externality.

All of these findings are proved to hold in the general model introduced here and some additional effects, mainly concerning city size, are identified as well.

In the second part of the paper, we extend the model to include the case of a polluting CBD. Then we investigate the pollution effect on land use. We find that pollution may produce a peculiar land rent pattern, especially near the CBD, where land rents may increase with distance for some range instead of decreasing everywhere. This possibility has been identified in earlier papers by Stull [15] and by Strotz and Wright [14]. We further show that net population density usually follows the land rent pattern—i.e., when land rent increases so does net population density, and when land rent decreases so does the net density. There is, however, a possible exception to this rule in our case. The unique phenomenon where land rent decreases with distance, while net density increases, can also occur in some range of the residential ring.

We then investigate the pollution distortion effect on the land use pattern with respect to the optimum land use pattern and discover that, contrary to the congestion and CBD size cases, the polluting

CBD distortion pattern is not unique. Indeed, there are several possible distortion patterns depending on the production function, dispersion function, utility function and utility level.

1. A TWO SECTOR MODEL WITH CONGESTION

1.1. The Assumption of the Model

Following traditional urban models [8–11], we assume a concentric town with a CBD of radius ϵ, which contains all production activity except housing, and a residential ring crossed by roads between ϵ and $\bar{U}$, where $\bar{U}$ is the radius of the town. The land outside the residential ring is agricultural land yielding a fixed rent of R_A.

The product produced in the CBD is an export good sold at a fixed price. Let $F(N, S)$ be the aggregate production function[2] of the industry where

N = Labor employed by the industry
S = Land used in production which is the CBD area.

Hence

$$S = (\tfrac{1}{2})\theta\epsilon^2 \quad 0 < \theta < 2\pi \tag{1}$$

θ being the section of land available for urban use.

We assume that F satisfies

$$\begin{aligned} &1.\ F_N, F_S \geqslant 0 \quad \text{for all } N \text{ and } S \\ &2.\ F_{NN}, F_{SS} < 0,\ F_{NS} > 0 \\ &3.\ F_{NN}F_{SS} - F_{NS}^2 > 0, \end{aligned} \tag{2}$$

The assumption that $F(\cdot)$ has decreasing returns to scale over the whole production scale is unrealistic, because the optimum city then consists of a single household. However, since the city formation is not the subject of this paper and since (2) is a reasonable assumption in the neighborhood of the equilibrium, we adopt it. For details on the process of city formation, see Henderson [4].

Households in the residential ring are assumed identical, with a utility function:

$$U(x) = U(b, z) \tag{3}$$

where $U(x)$ is the utility level of households residing x miles from the center, b is the quantity of housing available to the household, and z is the quantity of the composite commodity available to it.

Following Alonse [1, Chap. 2], housing is represented by the quan-

[2] $F(\cdot)$ is the value of the aggregate production of industry net of commuting and transportation costs within the CBD.

tity of land allocated to a household, i.e.,

$$b(x) = 1/a(x) \tag{4}$$

where $a(x)$ is the number of households per unit of residential land.

Each household contributes a single commuter to the CBD. The commuting is done via roads subject to congestion. Let $L(x)$ be the amount of land allocated for transportation at distance x from the center of the town, $T(x)$ be the number of commuters at distance x and $q(x)$ the commuting cost per traveler per mile at distance x. Note that $T(x)$ is the accumulated number of commuters who rseide between x and $\bar{U}$. Thus, we have

$$\begin{aligned} &1.\ d[T(x)]/dx = -a(x)(\theta x - L) \\ &2.\ T(\epsilon) = N \\ &3.\ T(\bar{U}) = 0. \end{aligned} \tag{5}$$

Congestion is given by

$$q(x) = q(T, L); \qquad \partial q/\partial T = q_T > 0; \qquad \partial q/\partial L = q_L < 0. \tag{6}$$

We assume the city is part of a system of cities in equilibrium, having the same homogeneous population, and that instantaneous and costless population migration is feasible. Therefore, the utility level of all individuals in all cities is equal to U_0. We further assume that there is an infinite supply of land at a rent R_A. We also assume farmers in the system are in equilibrium and thus have the same utility level, living in a single production consumption unit. We thus have

$$U(b(x), z(x)) = U_0 \quad \text{for } \epsilon \leqslant x \leqslant \bar{U}. \tag{7}$$

Consequently, we can see that the utility function reduces to a single indifferent curve.

Additionally, we assume that there are two sources of income in the city. One is the production process which takes place in the CBD. This process produces an export good which is sold outside for a fixed unit price. The second source of income comes from property shares owned by the inhabitants of the city. The pertinent property consists of shares in system-wide companies which own property in many cities in the system. Part of the property the company owns may be in the city where the company's shareholders reside. The assumption is that the residents of the city are not aware that their activity as residents may affect their income as share holders. Other assumptions are that the population is homogeneous with respect to skills, earning capacity and share holdings, which provide an equal yield to every resident in the system.

A system-wide equilibrium condition is that total income consumed in the system equals total produced income. If the production functions in all cities are equal, this condition reduces to a closed economy condition for each city. However, if we allow differences in production functions that explain differences in the size of the cities, transfer of income between cities is possible, and only the system-wide equilibrium condition holds. This condition determines the equilibrium utility, and since we investigate only a single unit of the system, we assume this as given and constant.

In this paper, we prove that laisser faire equilibrium does not insure Pareto optimality; and that Pareto optimality, which brings about an increase in total output and therefore an increase in the equilibrium utility level, requires government intervention at the city level.

1.2. Pareto Optimum and Market Equilibrium

Define the total surplus income of the city, SI, as the sum of surplus income in production, SIp, and surplus income in consumption, SIc: (SI = SIp + SIc). SIp and SIc are defined by

$$\mathrm{SIp} = F(N, X) - Nw - \tfrac{1}{2}\theta\epsilon^2 R_A \tag{8}$$

and

$$SI_c = Nw + Ny - 1/2\theta(\bar{U}^2 - \epsilon^2)R_A - \int_\epsilon^{\bar{U}} [T(x)q(x) + (\theta x - L)a(x)Z(x)]dx \tag{9}$$

Nw is industry's total payment for labor, the labor income of the population. Ny is the total non-labor income, with y the household yield from system-wide share holdings and w the wage rate. $(\frac{1}{2})\theta\epsilon^2 R_A$ is the alternative value of the land in the CBD and $(\frac{1}{2})\theta(\bar{U}^2 - \epsilon^2)R_A$ is the alternative value of the land in the residential ring. $\int_\epsilon^{\bar{U}} T(x)q(x)dx$ is the total expenditure on commuting in the residential ring and $\int_\epsilon^{\bar{U}} a(x)z(x)[\theta x - L(x)]dx$ is the total expenditure on the consumption of the composite commodity.

By (8) and (9), we get an expression for SI:

$$SI = SIp + SIc = F(N, S) + Ny - 1/2\theta\bar{U}^2 R_A - \int_\epsilon^{\bar{U}} [T(x)q(x) + a(x)z(x)(\theta x - L)]dx. \tag{10}$$

The system-wide equilibrium condition can now be expressed as follows: the total sum of SI over cities should be equal to the total sum of NY over cities.

A necessary and sufficient condition for a Pareto optimum is that SI is at its maximum level. If SI is not at its maximum level, the solution cannot be efficient since we can increase output without making anybody worse off, contrary to the assumption of Pareto optimality. If SI is at its maximum level, we can improve the welfare of an individual household only by reducing the welfare of another individual household. Thus, SI at its maximum level implies Pareto optimality. Consequently, necessary and sufficient conditions for maximum of SI are also necessary and sufficient conditions for Pareto optimality.

To derive the Pareto optimum conditions, we maximize SI in (10) subject to (5) and (7), with respect to N, S (or ϵ), $\bar{U}$, $a(x)$ [or $b(x)$], $z(x)$, and $L(x)$, where $T(x)$ is the state variable.

Define $H(x)$ as

$$H(x) = -T(x)q(x) + (Z(x) + \eta(x))a(x)(L(x) - \theta x) + \lambda(x) \times (U(X) - U_0) \quad (11)$$

where $\eta(x)$ is the auxiliary variable of (5.1) and $\lambda(x)$ is the Lagrange multiplier of (7). From now on, we refrain from using the index x unless the context so requires.

Necessary and sufficient conditions for the optimization problem are given by

$$-y - F_N - \eta(\epsilon) = 0 \quad (12)$$

$$-\theta\epsilon F_S + H(\epsilon) = 0 \quad (13)$$

$$H(\bar{U}) - \theta\bar{U}R_A = 0 \quad (14)$$

$$-\partial(Tq)/\partial L + (Z + \eta)a = 0 \quad (15)$$

$$(Z + \eta)(L - \theta X) + \lambda U_a = 0 \quad (16)$$

$$a(L - \theta X) + \lambda U_Z = 0 \quad (17)$$

$$\dot{\eta} - \partial(qT)/\partial T = 0. \quad (18)$$

Equations (12) and (13) give us the relationships between the residential sector and the industrial sector. Equation (14) establishes the relationship between the residential ring and the agricultural land further out. Equations (15)–(18) describe the relationships within the residential ring.[3]

$\eta(x)$ is the cost of absorbing a household in location x. From (12), we learn that at the CBD limit, it equals the marginal benefit of an

[3] We also assume that within the CBD land rents do not increase with distance from the center. Since commuting costs within the CBD have to be paid by industry, they act to decrease land rents in the CBD as distance from the CBD limit increases. The above assumption means that proximity to center has a greater advantage and consequently offsets the commuting advantage of distance from center.

additional laborer. By integrating (18), we get

$$\eta(x) = \eta(\epsilon) + \int_{\epsilon}^{x} [\partial(Tq)/\partial T]dx' \tag{19}$$

or

$$-\eta(x) = y + F_N - \int_{\epsilon}^{x} [\partial(Tq)/\partial T]dx'. \tag{20}$$

Equation (20) states that the costs of absorbing an additional household in location x should equal the marginal productivity of the household less the marginal commuting costs which it creates.

Designate by R the marginal productivity of land in transportation.

$$R = -\partial(Tq)/\partial L. \tag{21}$$

By substituting (20) and (21) into (15), we get

$$Rb + Z + \int_{\epsilon}^{x} [\partial(Tq)/\partial T]dx = F_N + y. \tag{22}$$

Equation (22) states that in the optimum the value of the marginal productivity of labor F_N plus non-labor income y should be equal to the cost of the marginal household's locating anywhere within the city borders, with the cost of a marginal household being the left-hand side of (22).

By dividing (16) by (17) and substituting (15) and (21) into the result, we get

$$R(x) = \frac{U_b(x)}{U_Z(x)} \quad \epsilon \leq X \leq \bar{U}. \tag{23}$$

By differentiating (7) and substituting in (23)

$$R(x) = -\frac{dz}{db}\cdot \tag{24}$$

By differentiating (22) with respect to x and substituting (24) in it, we get

$$R'b + \partial(Tq)/\partial T = 0. \tag{25}$$

This is the well-known equation which insures optimality with respect to location. Any change in land marginal productivity and costs of the composite commodity due to an infinitesimal movement in space is balanced by a change in transportation costs (see Muth [11]).

Substitute (21) into (16), and substitute the result in (12), and then calculate $H(\bar{U})$ from (11), noting that $T(\bar{U})$ and $L(\bar{U})$ vanish. We then get

$$R(\bar{U}) = R_A. \tag{26}$$

By substituting (21) into (15), this result and $x = \epsilon$ into (11) and finally into (13), we obtain

$$F_S = R(\epsilon) - \frac{1}{\theta\epsilon}(T(\epsilon)q(\epsilon) + L(\epsilon)R(\epsilon)). \tag{27}$$

Add to (21)–(27), the following equations:

$$F_N = w \tag{28}$$

$$\partial(TP)/\partial T = c(x), \tag{29}$$

where w is the wage rate paid to a worker at the CBD limit, and $c(x)\epsilon < x \leqslant \bar{u}$ is the commuting costs paid by an individual when travelling through x.

If we substitute $c(x)$ for $\partial(Tq)/\partial T)$ in (22) and (25), and substitute w for F_N, we get

$$Rb + z + \int_\epsilon^x c(x)dx = w + y \tag{22'}$$

which is the budget constraint of the individual household residing at x and (25) becomes

$$R'(x)b(x) + c(x) = 0, \tag{25'}$$

where $c(x)$ denotes commuting cost at x.

Define $v(x)$ as the bid rent curve of the industry. An equilibrium condition in the CBD is then

$$F_S(N, S) = v(\epsilon). \tag{30}$$

(30) holds only at the border of the CBD. Anywhere else in the CBD we have $F_S(N, S(x)) > v(x)$, where $S(x) = \frac{1}{2}\theta x^2$.

In a free market equilibrium (29′) holds instead of (29):

$$c(x) = q(x), \tag{29'}$$

the rationale being that $q(x)$ is the only cost the individual commuter faces. To achieve the optimum allocation, congestion tolls must be levied on the commuters so that (29) holds. For details, see Hochman [5], Oron *et al.* [12].

Another and more interesting observation is that in a free market equilibrium, we should have equality between residential and industrial bid rents in the CBD, that is, instead of (27), we should have:

$$v(\epsilon) = R(\epsilon). \tag{27'}$$

Therefore, to achieve the optimum via a price system, the government must subsidize land users in the CBD by the amount SU per land unit where SU is given by

$$SU = \frac{1}{\theta\epsilon}\,(T(\epsilon)q(\epsilon) + L(\epsilon)R(\epsilon)) > 0. \tag{31}$$

We thus establish that the optimum solution is described by (21)–(30) while for the free market equilibrium (22), (25), (27), and (29) are replaced by (22′), (25′), (27′), and (29′).

1.3. The Externality Effect of the CBD Size

The externality involved in (27) is different from most externalities. It is a positive externality caused by an expansion of the CBD area, while most externalities are of a negative sort. Since commuters have to pay for their commuting costs only to the CBD limit, an expansion of the CBD increases its circumference and therefore its accessibility. In other words, the total commuting distance of a given population and lot size decreases with an increase in the circumference of the CBD. Because in a free market situation industry is not compensated for this effect, although it is worthwhile from society's point of view to pay the industry to expand, the free market does not bring about this solution. (27′) rather than (27) holds in this case. Thus, the government has to subsidize industry by the amount in (31) per unit area to insure optimality. The term in (31) is the net gains in transportation costs due to a marginal expansion of the CBD, $T(\epsilon)q(\epsilon)$ being the marginal savings in total commuting costs and $L(\epsilon)R(\epsilon)$ the value of the marginal land previously involved in transportation which is no longer needed.

Note that the assumption that commuters pay for their commuting costs only to the CBD limit is essential for this externality: if the assumption were instead that commuters paid their full commuting cost, it would disappear. However, the assumption used here seems to be consistent with a competitive market. If in a free market laborers were not compensated for working farther away from the CBD limit, they would concentrate on the CBD limit and there would be excess demand for labor inside the CBD and excess supply of labor on the border. Thus, labor cost must be higher inside the CBD as compared to the CBD limit to compensate for the excess commuting cost, which is precisely the assumption made here.

We can now try to determine the effect of a subsidy on land use, the rent function and city size, and thereby establish the distortions caused by free competition.

The initial effect of the subsidy is straightforward. Under perfect competition, it increases the radius of the CBD because land rents facing the industry decrease. From (3.2), we learn that for fixed N, F_N increases with ϵ, and therefore total income of an individual increases (see (22)). Commuting costs per family decrease due to the expansion of the CBD. Since utility is constant, the surplus income per household due to these two effects is absorbed by an increase in land rents at each location. This increase causes households to reduce lot size and substitute for it by increasing consumption of the composite commodity z. Roads also become narrower. Hence, at each location we have an increase in both density and congestion. From (26) we learn that this leads to an increase in the absolute value of the derivative of the rent function. Nevertheless, the increase in density and congestion can never cause an individual to end up with less income for housing and consumption than before. To prove this contention, note that if commuting costs did at any location leave the household with less income for housing and consumption than in the previous case, then at this location land rents must decrease below their previous level to compensate the consumers, and leave them at the same utility level. We therefore assume that there is a location x_1, in which $R^*(x_1) < R_1(x_1)$, and then prove a contradiction. The index * designates the optimum case and the index 1 designates the free market case. The existence of x_1 implies the existence of x_0, where $R^*(x_0) = R_1(x_0)$, $x_0 < x_1$. This means that commuting costs from x_0 to ϵ^* leave the household with the same income for consumption and housing as before. Consider a household in x_1. Commuting from ϵ to x_0 leaves him with the same income in x_0 as before. However, to commute from x_0 to x_1 costs less in the optimal case since we assumed $R^*(x_1) < R_1(x_1)$, and the broader roads and less dense population decrease commuting costs. Consequently, a household in the optimum case ends up in x_1 with more income for housing and consumption than in the case of no government interference. Therefore, $R^*(x_1)$ should be greater than $R_1(x_1)$, which inequality is a contradiction.

To sum up, we have:

$$R^*(x) \geqslant R_1(x) \quad 0 \leqslant x \leqslant \bar{U}^*$$

$$|dR^*(x)/dx| > |dR_1(x)/dx| \quad \epsilon^* \leqslant x \leqslant \bar{U}^*$$

and if there is an x_0 which fulfills $R^*(x_0) = R_1(x_0)$, then $R^*(x) = R_1(x)$ for every $x > x_0$.

In addition, we established

$$\bar{U}^* \geqslant \bar{U}_1$$

$$N^* > N_1$$

$$S^* > S_1$$

and $F(N^*, S^*) > F(N_1, S_1)$. All the above results are depicted in Fig. 1.

It is interesting to see that in this case, we could achieve the same result by taxing residential land use instead of subsidizing industrial land use. However, when doing this, we should make certain that the tax in each location x does not exceed $R(x) - R_A$, for were it to do so, city radius and city size would decrease below their optimal sizes. In the case of subsidies, however, it is relatively simple to calculate the optimal subsidy.

1.4. Congestion Tolls Versus Free Market Equilibrium

This subject has already been investigated in several articles (see for example, Hochman [5], Oror *et al.* [12]), and we therefore discuss it briefly with only a few extensions.

Congestion tolls in this model are given by (32):

$$\mathrm{Toll}(x) = T\partial q(T, L)/\partial T. \tag{32}$$

Note that if Tq is linear homogeneous in T and L, then the tolls equal factor costs. See, for example, Hochman [5]. Actual data (see Vickery [17]), support this assumption of linear homogeneity. It is utilized in several recent papers dealing with congestion such as, *inter alia*, Mills and de Feranti [10], Solow and Vickery [13], Hochman and Pines [6].

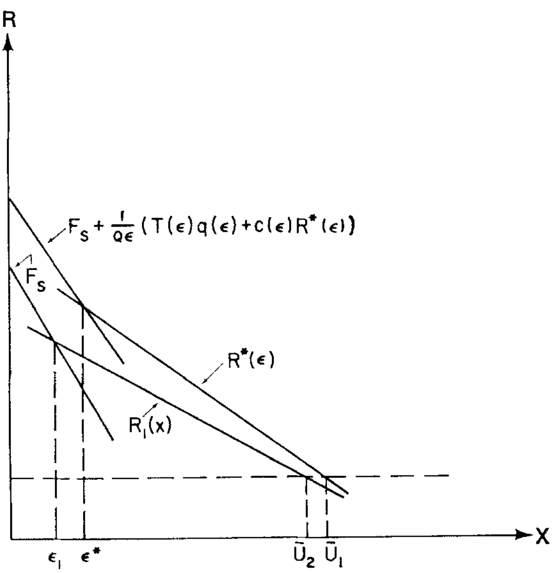


FIGURE 1

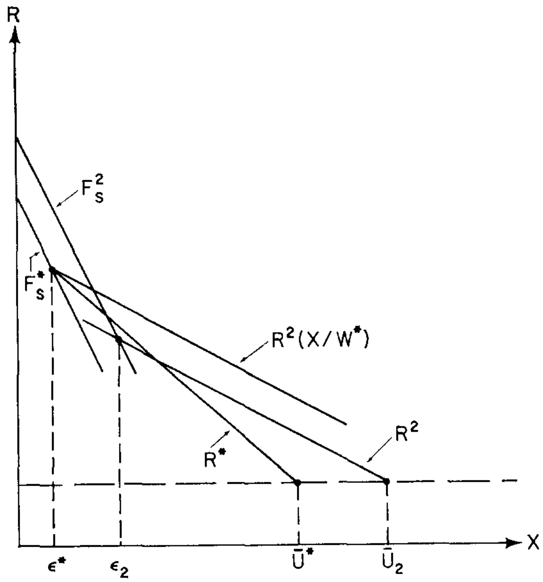


FIGURE 2

Let us now investigate how decreasing $c(x)$ affects land use. Note that in this case, the direction of investigation is from the optimum to the free market equilibrium. We continue to use * for the optimum, using the index 2 for the case without congestion tolls. It is clear from (7) and (22′), that $w + y$ alone determines $z(\epsilon)$, $b(\epsilon)$, and $R(\epsilon)$. Since initially w is not changed by a change in $c(x)$, all those variables remain unchanged when $c(x)$ decreases. From (25′), it is clear that $R'(\epsilon)$, however, increases. Therefore, in a neighborhood of ϵ, $\{R_2(x)/_{w^*} > R^*(x);\ \epsilon < x < \epsilon + \delta\}$. Clearly, $R_2(x)/_{w^*}$ must always be greater than or equal to $R^*(x)$. Since density increases with rent, this condition implies that the net density of the population everywhere is higher in the reduced $c(x)$ case for a given w^*. It also follows from (21) that land will be used more intensively in transportation as well, and therefore that gross density will increase as well. In Fig. 2, we see that this implies that N and $\bar{U}$ must increase. But if N increases, (28) and (2.2) imply that w decreases; hence, $w_2 < w^*$. The bid rent curve for w_2, $R_2(x)$ therefore passes below and parallel to $R_2(x)/_{w^*}$ and crosses $R^*(x)$ somewhere, so that N_2 is still larger than N^* (otherwise w does not decrease). Hence, $F_S(N_2)$ will be above $F_S(N^*)$. We may thus conclude, using Fig. 2, that when $c(x)$ decreases, the CBD limit, the radius of the city and the city size increase. The density decreases in the center of town and increases in the suburbs.

1.5. *Distribution of Income Between Sectors*

The question is to whom SI goes in our model. Part of the answer is provided by Stull. In his model, the developer absorbs his net gains through land rents. The intuitive explanation for how the developer can, by using land rents, absorb the total SI is straightforward. Since land is the only fixed factor in both the CBD and the residential ring, the landlord will be able, through competition between plants in the CBD and between households in the residential ring, to absorb the net gains of industry and the excess income of the resident-laborers. The same applies in our case, provided land is the only fixed factor. Then land rents in the CBD absorb SIp and land rents in the residential ring absorb SIc. If, however, land is not the only fixed production factor, then SIp is divided between land rent and rents to other fixed production factors according to the following rule: if the quantity of the other fixed factors is smaller than it should be in the long-run equilibrium implied by the long-run costs of the factor, then $F(\cdot)$ has short run diseconomies of scale, and SIp is divided between land rents and rents to the fixed factors. If the quantity of the fixed production factor is greater than it should be in the long run, then $F(\cdot)$ has economies of scale, and land rents will exceed SIp, with additional payments coming from short-run losses to the other fixed factor, in which case we say that rents to the other fixed factors are negative.

Government intervention in economic activity in the city is represented by government investment. Taxes are considered negative investment and subsidies and payments to economic participants are positive investment. Then we have[4]

$$\text{SI} + \text{net government investment} = \text{total land rents} + \text{rents to other fixed factors.}$$

2. ANALYSIS OF A POLLUTING CBD

2.1. *An Extension of the Model Assumptions*

We now extend the model to include the case where the industry in the CBD produces as a by-product pollution which then disperses over the residential ring and causes welfare losses. We assume away any externalities within the industry. Let K be the value of the export product and S and N be space and labor as before. p is the quantity of pollution produced by S and N as a byproduct of K. We assume that p is generated in the center of town or, rather, that p is the amount of pollution in the center of town equivalent in its effect on the resi-

[4] A formal proof can be worked out if we assume long run linear homogeneous production function at each location in the CBD. In the residential ring, a proof can be worked out by calculating $\int_\epsilon^{\bar{u}} R(x)x\,dx$ from Eq. (22′) in the text.

dential ring to the pollution produced over S by using N labor units in the production of K. We extend the definition of $F(\cdot)$ to include pollution p as a byproduct as follows:

$$
\begin{aligned}
&(1)\ K = F(N, S, p)\\
&(2)\ F_N, F_S, F_P \geqslant 0 \text{ for all } N \text{ and } S, \text{ and for } P \leqslant \bar{P}(N, S)\\
&(3)\ F_{NN}, F_{SS}, F_{PP} < 0 \qquad (33)\\
&(4)\ F_{NS} > 0,\ F_{Np} \lesseqgtr 0,\ F_{Sp} = 0\\
&(5)\ F(\cdot) \text{ is a convex function in all variables where:}
\end{aligned}
$$

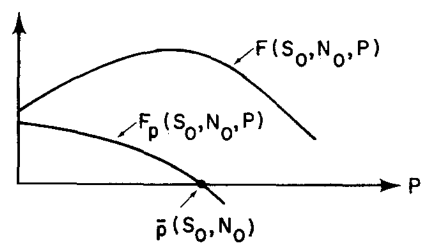
$$\bar{P}(N, S) \text{ fulfills } F_p(N, S, \bar{P}(N, S)) = 0 \quad \text{and} \quad \partial\bar{P}(N, S)/\partial S \geqslant 0.$$

(33.2) and (33.4) need some elaboration. $F_p \geqslant 0$ stipulated that a range exists where the industry can, by increasing the amount of pollution discharged, increase its revenue by saving on costs of pollution control, substituting cheaper and more polluting raw materials (for instance, by shifting from oil to coal) or by using cheaper but more polluting production processes (e.g., eliminating recycling of processing water or using lower stacks).[5] In (33.4) we assumed that labor and space are complementary factors, and since we have no particular reason to assume complementarity or substitution between space and pollution, we assume independence. Between labor and pollution, we will assume alternatively complementarity and substitution, investigating both cases.[6]

The pollution generated in the center of town is dispersed over the city through a dispersion function[7] where $g(\cdot)$ fulfills

$$
\begin{aligned}
g = g(x, p) \quad g(0, p) = \mathrm{p}, \quad g(x, 0) = 0\\
\partial g/\partial x < 0 \quad \partial g/\partial p > 0.
\end{aligned} \qquad (34)
$$

[5] For a similar approach to the question of pollution, see Tolley [16]. It can be more easily described graphically by the following figure. Let S_0 and N_0 be given, then



We assume no externalities within the industry, i.e., no indirect damages by pollution to industry.

[6] Note that we could assume and investigate a greater variety of cases. However, the concommitant is a greater variety of solutions, and we have already burdened the reader enough.

[7] Consequently we assume that pollution disperses symmetrically over all the urban region which, of course, is a simplified assumption.

Thus, $g(x, p)$ is the concentration of pollution at location x as a result of a pollution level p, produced in the center of town. Let $z(x)$ be the composite consumption commodity. Then

$$z(x) = \phi(\mu, g(x, p)), \tag{35}$$

where μ is the vector of market consumption goods with a given constant price and g is the pollution concentration at location x caused by emission generated at the center of town, of intensity p. The concept of a production consumption function was first introduced by Gary S. Becker. It can be shown that by assuming a utility function $U(b, z)$ with $z = \phi(\mu, g)$, we do not lose any generality over the traditional case where pollution is introduced directly into the utility function, i.e., $U = U(b, z, g)$. (For details, see [7].) We assume $\partial\phi/\partial\mu > 0$ and $\partial\phi/\partial g < 0$. Hence, $\partial\phi/\partial x = (\partial\phi/\partial g)(\partial g/\partial x) > 0$ and $\partial\phi/\partial p = (\partial\phi/\partial g)(\partial g/\partial p) < 0$. Let us consider a unit of z, and designate by π the cost of a unit of z. For a given level of pollution (p) produced at the center of town, we have $g(p, x_1) > g(p, x_2)$, for $x_1 < x_2$. Thus, to produce the same unit of z, we have to use more μ in x_1 than in x_2. Since the cost of g is zero, the cost of z is the cost of the μ, and since $\mu(x_1) > \mu(x_2)$, we have $\pi(p, x_1) > \pi(p, x_2)$, where $\pi(p, x)$ is the cost of z at location x, with p the level of pollution at the center of town. We also know that g increases with p. Thus, at a given location if $p_1 < p_2$, then $g(x, p_1) < g(x, p_2)$, and therefore, we need less of μ to produce a unit of z at x with p_1 than with p_2, i.e., $\pi(x, p_1) < \pi(x, p_2)$. We can summarize the discussion by the following quantitative statements about the price function:

$$\begin{gathered}\pi(p, x) \geqslant 1, \quad \pi(0, x) = 1 \\ \partial\pi/\partial p = \pi_p > 0, \quad \partial\pi/\partial x = \pi_x < 0 \\ \partial\pi^2/\partial p\partial x^8 < 0 \quad \text{for all } x, p > 0.\end{gathered} \tag{36}$$

The intuitive rationale behind this assumption is that pollution causes physical damage to consumption goods by increasing deterioration, wear and tear and that accordingly, more cleaning and maintenance are needed. Pollution damages health and hence increases the needs for medical treatment and more frequent recreation further away from home. More things need storage and storage is more expensive. In sum, pollution increases the cost to the household per unit of goods consumed as described in (36).

[8] This is a convenient assumption and the opposite assumption could serve just as well.

2.2. Necessary Conditions for Optimality and Equilibrium

Equations (21)–(32) with (22′), (26′), (28′), and (30′) apply here too, with a few changes, instead of (22′), (23) and (25), we have;

$$\pi z + Rb + \int_{\epsilon}^{x} c(x)dx = w + y \tag{22}$$

$$R(x)/\pi(P, x) = U_b/U_z \tag{23}$$

$$c(x) + \pi_x z + R'b = 0 \tag{25}$$

In addition, we will have

$$F_p = A, \tag{37}$$

where A is the cost of a unit of pollution. In the free market case, we have

$$A = 0. \tag{38}$$

In the optimum, we have

$$A = \int_{\epsilon}^{\bar{U}} (\theta x - L(x))z(x)\pi_p(p, x)a(x)dx. \tag{38'}$$

The right hand side of (38′) is the marginal damage to the population caused by an additional unit of emissions. And, if we set A equal to this damage, say through pollution taxes, we achieve an optimal solution.

From (25), we now have

$$R'(b) = -a[\pi_x z + c(x)]. \tag{39}$$

$c(x)$ is always positive, while $\pi_x z$ is always negative. Since z and b are functions of R/π alone, it may be that for $|\pi_x|$ big enough, we get positive or non-negative values for R'. In the range where R increases with distance, so will $[R/\pi]$. Consequently, we may find an increase in net density, $a(x)$, when moving away from the CBD. Note, however, that if π is constant, $[R/\pi]$, is always negative. Since π decreases with distance, R/π can increase even if R decreases so that net density will again increase. The behavior of $R(x)$ and $a(x)$ depends on the relationship between $|\pi_x z|$ and $|c(x)|$, both of which decrease with distance. Similar results with regard only to the rent function can be found in Stull [15] and Strotz *et al.* [14].

In Fig. 3, some of these possible relations are described. Some attention should be given to a unique phenomenon just described here, the possibility of an *increase* in net density while rent is *decreasing*. While such a phenomenon has been observed empirically, this is its first theoretical explanation.

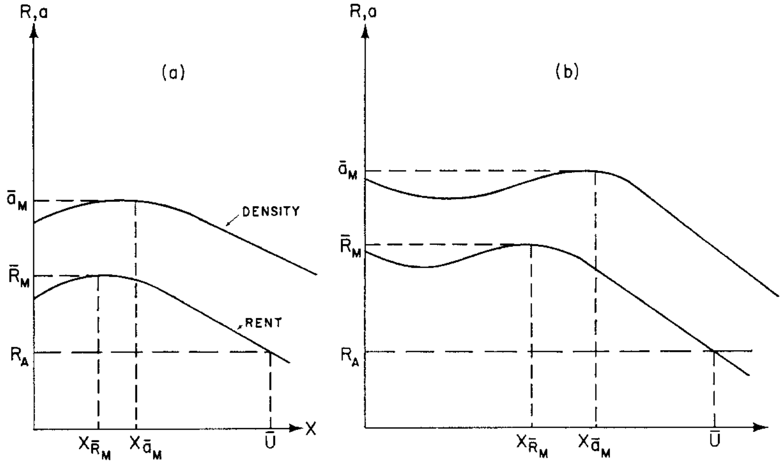


FIGURE 3

We should also pay some attention to the measurement of the tax pertinent to this case. Assume that π is a function of the form

$$\pi(p, x) = D(x)p^{\alpha} + 1, \tag{40}$$

where $D(x)$ is some function of x, then $\pi_p = (\alpha/p)(\pi - 1)$. We can now write (38) as

$$A = \frac{\alpha}{p}\left[\int_{\epsilon}^{\bar{U}} (\theta x - L)a\pi z dx - \int_{\epsilon}^{\bar{U}} (\theta x - L)az dx\right]. \tag{41}$$

The first term in the brackets is the total expenditure on the composite good. The second term is the total expenditure on the composite good if no pollution exists. The difference between the two is the total damage caused by pollution. Consequently, we find that the total tax Ap should be proportional to the total damages, with α the proportion coefficient. The measurement of p is an engineering problem, while estimation of the total damages is possible in principle. The problem is the estimation of α. There is no known way of estimating it, other than experimenting with different pollution levels and observing changes in their associated long run land rents. Such experimentation necessarily takes a long time and it is impossible to assume constant utility levels, technology and preferences during such a long period.

Accordingly, the best policy for a policymaker to adopt is probably to assume $\alpha = 1$. Note that in this case, the pollution tax should equal the pollution damages.

TABLE 1

The Effects[a] of an Increase in Pollution on City Parameters

Specification of the system / Variables affected	$F_{NP} < 0$ (1)	$F_{NP} > 0$: $\Delta(\mathrm{I}(\epsilon))/\Delta P > 0$ (2)	$F_{NP} > 0$: $\Delta(\mathrm{I}(\epsilon))/\Delta P < 0$; $\Delta(\mathrm{I}(\bar{U}^*))/\Delta P < 0$; $\dfrac{\Delta[I(U^*) - \frac{1}{\pi}\int_{\epsilon}^{\bar{U}^*} c(x)dx]}{\Delta P} > 0$ (3)	$F_{NP} > 0$: $\Delta(\mathrm{I}(\epsilon))/\Delta P < 0$; $\Delta(\mathrm{I}(\bar{U}^*))/\Delta P < 0$; $\dfrac{\Delta[I(U^*) - \frac{1}{\pi}\int_{\epsilon}^{\bar{U}^*} c(x)dx]}{\Delta P} < 0$ (4)	$F_{NP} > 0$: $\Delta(\mathrm{I}(\epsilon))/\Delta P < 0$; $\Delta(\mathrm{I}(U^*))/\Delta P > 0$ (5) u·d[b] if i	(5) u·d[b] if d
City size (population)	d	i	d	d	i	d
Radius of town	d	i	i	d	i	i
Radius of CBD	ud	ud	ud	ud	i	ud
Wage rate—W	d	i	i	i	i	i
Rent near $\epsilon^*(R)$	d	i	ud	ud	i	d
Real rent near $\epsilon^*(R/\pi)$	d	i	d	d	i	d
Rent near $\bar{U}^*$	d	i	i	ud	i	i
Real rent near $\bar{U}^*$	d	i	i	d	i	i

[a] There are three possible characteristics of the change: i—increase, d—decrease, ud—undetermined.

[b] In this case the given specification was not enough to separate possible results due to two opposing effects: decrease of real income in the CBD limit and increase of net real income near the radius of town. It was, therefore, characterized using the direction of the change in city size as specifications.

2.3. The Externality Effect of Pollution Regulation in Point Source Pollution

In this section, we examine the distortion of the competitive equilibrium land use patterns as compared to the optimum allocation pattern. The method we use to evaluate the distortion effect on the different parameters is essentially the same one we used in Sec. 1. We present in detail only one representative case and sum up the rest of the cases in Table 1. Note that pollution in the competitive case is always greater than in the optimal case. Thus, we have only to determine the effect of an exogenous increase in pollution on the different parameters.

The case we trace out in detail is that in which $F_{NP} < 0$, i.e., pollution and labor are substitutes.

For a constant N, an increase in p causes F_N to decrease. This causes both w and $R(x)$ to decrease. Accordingly, $b(x)$ and $L(x)$ increase and $z(x)$ decreases. A decline in $R(x)$ for every x also implies, as is shown in Fig. 4, that the radius of the city decreases.

The shrinking of the residential ring and the increase in consumption of land (for residential and transportation use) per family mean that N must decrease. This counteracts the initial effect of the increase in pollution on F_N since F_{NN} is negative, and thus a stable equilibrium is achieved. It also causes a downward shift in $F_S(S)$. The final equilibrium position is one of more pollution, a smaller size and a smaller radius of the city than are optimum. People consume everywhere more land and less of the composite commodity, and transportation is produced with more land. The situation is described in Fig. 4.

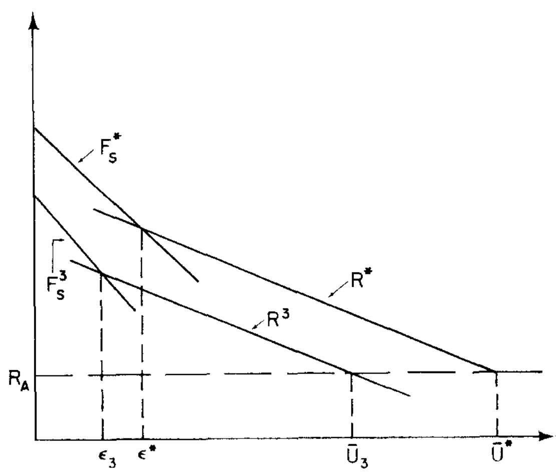


FIGURE 4

The results of the rest of the cases are described in Table 1. All these cases assume that pollution and labor are complements in production. Define $I(x)$, individual real income in location x, as

$$I(x) = (w + y)/\pi(x, P). \tag{42}$$

Then case (2) in the table is characterized by an increase in real income near the CBD, and hence in the rest of the city, due to an increase in pollution. The increase comes from an increase in the wage rate which is not offset by an increase in the price of the composite commodity.

In the rest of the cases, we have a decrease in real income near the CBD. In cases 3 and 4, we also have a decrease in real income near the limit of the city. In case 5, which also subdivides into two cases, we have a decrease in real income near the CBD limit but an increase near the border of town, i.e., an effective dispersion of pollution with distance.

Cases (3) and (4) differ in the effect of the pollution near the city limit on real income net of real commuting costs, i.e., on the real income left over for consumption and housing.

The variety of solutions teaches us that we cannot hope to learn from one city how to behave and what to expect in another city. Cities differ from one another in production functions, dispersion functions (different climates, sizes, wind direction, etc.), utility functions (different types of populations) and real wage rate (utility levels). Therefore, the relationship between optimal and competitive allocations varies from city to city.

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